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A field-like property of finite rings

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ABSTRACT

Let χ be a fixed non-principal character of the additive group $\mathbf{F}^+$ of a finite field $\mathbf{F}$. Any character ω of $\mathbf{F}^+$ can be found by the rule $\omega(x)=\chi(ax)$ for a well-chosen $a\in\mathbf{F}$. In this paper we investigate for which finite rings an analogous property holds.

1. INTRODUCTION

The need for a property of rings as it is described in the abstract arose during our investigation of cyclotomic schemes over finite rings; these are association schemes "induced" by a group of units of a finite ring. We call such rings "admissible rings". (For cyclotomic schemes the reader is referred to [1,3].)

As the notion of admissibility seems to be an interesting one on its own we investigate at first the notion in detail, whereas in the second part of this paper we consider consecutively the following classes of rings with identity: rings with a few minimal one-sided ideals, semi-simple rings, local rings, commutative rings.

2. PRELIMINARIES

In this paper $\mathbf{R}$ denotes a finite ring with υ elements. Most of the time we suppose that $\mathbf{R}$ has an identity $1\neq 0$; only on the first few pages, unless otherwise stated, $\mathbf{R}$ need not necessarily have an identity. $\gamma(\mathbf{R})$ denotes the characteristic of $\mathbf{R}$. For most of the ring theory used in this paper we refer to [5].

If $x\in\mathbf{R}$ then (x) denotes the (two-sided) ideal generated by x, $\mathbf{R}x$ denotes the left ideal generated by x and $x\mathbf{R}$ denotes the right ideal generated by x. The ideals $\mathbf{R}$ and (0) are called *trivial*, the other ones *proper*.

The (*Jacobson*) *radical* of the ring $\mathbf{R}$, denoted by $\mathbf{Rad(R)}$ or by $\mathbf{Rad}$, is the intersection of all maximal right (left) ideals of $\mathbf{R}$.

$\mathbf{R}$ is called *simple* if it does not contain any proper two-sided ideals, *semi-simple* if $\mathbf{Rad(R)}=(0)$ and *local* if $\mathbf{R}/\mathbf{Rad(R)}$ is a (finite) field.

$\mathbf{R}^+$ is the additive group of the ring $\mathbf{R}$. $\mathbf{U}$ or $\mathbf{U(R)}$ will denote the (multiplicative) group of all units in $\mathbf{R}$.

By $\mathbf{N}$ we denote the set of the natural numbers, by $\mathbf{Z}$ the ring of the integers, by $\mathbf{Z}_\upsilon$ the ring of integers modulo υ and by $\mathbf{C}$ the field of the complex numbers.

In this paper we shall give several examples, most of the time using finite rings of the form, say, $\mathbf{Z}_\upsilon[x_1,x_2,\dots,x_m]/(f_1,\dots,f_k)$ and $f_1,\dots,f_k$ are polynomials in the unknowns $x_1,x_2,\dots,x_m$ which are supposed to be *non-commuting*.

For any subset $\mathbf{B}$ of $\mathbf{R}$ the sets

$$\mathbf{Ann}^l(\mathbf{B})=\{x\in\mathbf{R}\mid xb=0 \text{ for all } b\in\mathbf{B}\}$$

and

$$\mathbf{Ann}^r(\mathbf{B})=\{x\in\mathbf{R}\mid bx=0 \text{ for all } b\in\mathbf{B}\}$$

are called the *left* and the *right annihilators* of $\mathbf{B}$ in $\mathbf{R}$, respectively.

The notion of the "socle" of a ring (due to Dieudonné [2]) is important in this paper.

DEFINITION 2.1. The sum of all minimal right (left) ideals in a ring $\mathbf{R}$ is called the *right* (*left*) *socle* of $\mathbf{R}$ and will be denoted by $\mathbf{S}^r(\mathbf{R})$ or $\mathbf{S}^r$ ($\mathbf{S}^l(\mathbf{R})$ or $\mathbf{S}^l$).

If $\mathbf{R}$ is commutative then we call $\mathbf{S}^r(\mathbf{R})$ the *socle* of $\mathbf{R}$ and denote it by $\mathbf{S(R)}$ or by $\mathbf{S}$.

We note that several statements for, say, the left socle can also be formulated for the right socle of the ring.

It is well-known (see for instance [2,4]) that the left socle of a ring is an ideal.

If $\alpha:\mathbf{R}\to\mathbf{R}$ is an *anti-automorphism* (that is $\alpha(x+y)=\alpha(x)+\alpha(y)$ and $\alpha(xy)=\alpha(y)\alpha(x)$) then clearly $\alpha(\mathbf{S}^l)=\mathbf{S}^r$.

For later use we prove the following lemmas.

LEMMA 2.2. *For any finite ring* $\mathbf{R}$ *with identity we have* $\mathbf{S}^r\cdot\mathbf{Rad}=(0)$.

PROOF. Let $x\mathbf{R}$ be a minimal right ideal and let $z\in\mathbf{Rad(R)}$. Then $xz\mathbf{R}\subset x\mathbf{R}$ and if $xz\neq 0$ then $xz\mathbf{R}=x\mathbf{R}$. So there is a $z_0\in\mathbf{R}$ such that $xzz_0=x$, implying

$x(zz_0-1)=0$. Because $zz_0 \in \mathbf{Rad(R)}$ we have $zz_0-1 \in \mathbf{U(R)}$ and we find $x=0$, a contradiction. So $xz=0$.

Because every element of $\mathbf{S}^r$ is the direct sum of generators of minimal right ideals of $\mathbf{R}$ the assertion of the lemma follows readily. □

LEMMA 2.3. *For any finite ring* $\mathbf{R}$ *with identity we have*

$$(\mathbf{Rad} \cap \mathbf{S}^r) \cdot \mathbf{S}^r = (0).$$

PROOF. In this proof $z \in \mathbf{R}$ has the property that $z\mathbf{R}$ is a minimal nilpotent right ideal, which implies $z\mathbf{R} \subset \mathbf{Rad} \cap \mathbf{S}^r$ and so $z \in \mathbf{Rad} \cap \mathbf{S}^r$.

First let $x \in \mathbf{R}$ have the property that $x\mathbf{R}$ is a minimal nilpotent right ideal. $x\mathbf{R}$ is nilpotent implies $x \in \mathbf{Rad}$. Hence by lemma 2.2 we have $zx=0$.

Now let $x \in \mathbf{R}$ have the property that $x\mathbf{R}$ is a minimal idempotent right ideal. If $zx \neq 0$ then $zx\mathbf{R}$ is a minimal right nilpotent ideal isomorphic, in the sense of Dieudonné [2], to $x\mathbf{R}$. Therefore $zx\mathbf{R} \cdot x\mathbf{R}$ is an idempotent right ideal, but

$$z(x\mathbf{R}x\mathbf{R}) \subset z\mathbf{R} \subset \mathbf{Rad},$$

a contradiction. So $zx=0$ in this case, too. □

REMARK 2.4. We note the following.

1. A ring $\mathbf{R}$ with identity is semi-simple if and only if $\mathbf{R}=\mathbf{S}^r$ (cf. theorem VIII.1 in [5] and lemma 2.2).
2. If the ring $\mathbf{R}$ with identity is not semi-simple then $\mathbf{S}^r$, as a ring, has no identity (cf. lemma 2.3).
3. In general $\mathbf{S}^l \neq \mathbf{S}^r$. $\mathbf{GF}(q)[x,y]/(xy, x^2-yx, y^2-yx)$ is a non-commutative ring for which $\mathbf{S}^l=\mathbf{S}^r$ holds, however.
4. The ring $\mathbf{R}_0$ of the upper triangular (2×2)-matrices over $\mathbf{GF}(q)$ is an example of a non-commutative ring for which $\mathbf{S}^l \neq \mathbf{S}^r$. Clearly

$$\mathbf{S}^r = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbf{GF}(q) \right\}$$

whereas

$$\mathbf{S}^l = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, a, b \in \mathbf{GF}(q) \right\}.$$

Notice that

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$$

is an anti-automorphism of $\mathbf{R}_0$. Obviously $\alpha(\mathbf{S}^l)=\mathbf{S}^r$. ◇

3. ADMISSIBILITY

Let $\mathbf{G}$ be an abelian group of finite order. A *character* of $\mathbf{G}$ is a function $\chi : \mathbf{G} \to \mathbf{C}$ such that

1. $\chi(g+h)=\chi(g)\chi(h)$ for all $g,h\in\mathbf{G}$,
2. $\chi(g)\neq 0$ for some $g\in\mathbf{G}$.

The character χ of $\mathbf{G}$ such that $\chi(g)=1$ for all $g\in\mathbf{G}$ is called the *principal* character.

By $\mathbf{Ker}(\chi)$ we denote the set $\{x \mid \chi(x)=1\}$: the *kernel* of χ.

For any character χ of the additive (abelian) group $\mathbf{R}^+$ of the ring $\mathbf{R}$ (that is for an additive character of $\mathbf{R}$) we define for $a\in\mathbf{R}$ the following characters: ${}^{(a)}\chi$ by ${}^{(a)}\chi(x)=\chi(ax)$ for all $x\in\mathbf{R}^+$ and $\chi^{(a)}$ by $\chi^{(a)}(x)=\chi(xa)$ for all $x\in\mathbf{R}^+$.

Of course, ${}^{(0)}\chi=\chi^{(0)}$ is the principal character and if $\mathbf{R}$ has an identity ${}^{(1)}\chi=\chi^{(1)}=\chi$.

DEFINITION 3.1. Let χ be any character of $\mathbf{R}^+$. Let Ω^l_χ be the subgroup of the character group Ω of $\mathbf{R}^+$ defined by $\Omega^l_\chi=\{{}^{(a)}\chi \mid a\in\mathbf{R}\}$.

We say χ is *left admissible* if $\Omega^l_\chi=\Omega$; the notion of *right admissible* character is defined analogously. A character which is both left admissible and right admissible will be called *admissible.*

A ring $\mathbf{R}$ which has at least one left admissible character will be called a *left admissible* ring; in the same way a *right admissible* ring is defined.

A ring $\mathbf{R}$ that is both left and right admissible will be called an *admissible* ring.

To avoid unnecessarily complicated formulations, we shall state, where appropriate, our results only for left admissible rings.

Finite fields are admissible rings, but the next example shows that not all rings are admissible.

EXAMPLE 3.2. Let $\mathbf{R}_1=\mathbf{GF}(2)[x,y]/(x^2,y^2,xy,xy-yx)$. $\mathbf{R}_1$ is a commutative, local, non-principal ideal ring with 4 non-trivial ideals. The minimal ideals are $\{0,x\}$, $\{0,y\}$ and $\{0,x+y\}$, whereas the maximal ideal is equal to the socle $\mathbf{S}=\{0,x,y,x+y\}$.

$\mathbf{R}_1^+$, considered as a $\mathbf{GF}(2)$-vectorspace, has a basis $(1,x,y)$. The values of any character of $\mathbf{R}_1^+$ are completely determined by the values on this basis. Investigating the characters it is easy to show that $\mathbf{R}_1^+$ has no admissible characters, and so $\mathbf{R}_1$ is not admissible. For example, if $\chi(1)=-\chi(x)=\chi(y)=1$ then ${}^{(1)}\chi={}^{(1+y)}\chi=\chi$, which shows that χ is not admissible, etc. ◇

We start our investigation of admissibility by developing several other characterizations of this notion.

Notice that if one uses theorem 3.3 to study *left* admissibility one has to consider *principal right* ideals

THEOREM 3.3. *Let* $\mathbf{R}$ *be a ring such that* $\mathbf{Ann}^l(\mathbf{R})=\{0\}$ *then* $\mathbf{R}$ *is left admissible if and only if there is a subgroup* $\mathbf{K}$ *of* $\mathbf{R}^+$ *such that*

1. *the quotient group* $\mathbf{R}^+/\mathbf{K}$ *is cyclic and*
2. *the only principal right ideal contained in* $\mathbf{K}$ *is the ideal* (0).

If **R** *is a left admissible ring then for the group* **K** *one can take the kernel of a left admissible character.*

PROOF. Suppose that **R** is a left admissible ring, then take for χ a left admissible character and let $\mathbf{K}=\mathbf{Ker}(\chi)$. Then, clearly, the quotient group $\mathbf{R}^+/\mathbf{K}$ is cyclic.

Suppose there is a proper right ideal **I** contained in **K**. Let a be a non-zero element of **I**, then $ax\in\mathbf{I}$ for all $x\in\mathbf{R}$ and because $\mathbf{I}\subset\mathbf{K}$ we find ${}^{(a)}\chi(x)=\chi(ax)=1$ for all $x\in\mathbf{R}$. Hence ${}^{(a)}\chi={}^{(0)}\chi$. But this implies that $|\Omega^l_\chi|<|\mathbf{R}^+|$, which is obviously a contradiction, since Ω is isomorphic to $\mathbf{R}^+$.

Assume, now the other way around, that **K** is a subgroup of $\mathbf{R}^+$ having the properties 1. and 2. mentioned in the theorem.

Under these conditions **K** is the kernel of a character χ of $\mathbf{R}^+$. Suppose for different elements a and b of $\mathbf{R}^+$ we have ${}^{(a)}\chi={}^{(b)}\chi$. Then ${}^{(a)}\chi(x)={}^{(b)}\chi(x)$ for all $x\in\mathbf{R}^+$, implying $\chi((a-b)x)=1$ for all $x\in\mathbf{R}^+$. Hence the (principal) right ideal $(a-b)\mathbf{R}\neq(0)$ belongs to **K**, which yields (according to 2.) a contradiction. □

LEMMA 3.4. *If* **R** *is a ring without identity and* $\mathbf{Ann}^l(\mathbf{R})=\{0\}$ *then for every* $a\in\mathbf{R}$ *there is an element* $b\in\mathbf{R}\setminus\{0\}$ *such that* $ab=0$.

PROOF. Let $\mathbf{R}_0=\mathbf{R}\setminus\{0\}=\{x_1,x_2,\dots,x_{\upsilon-1}\}$. Suppose there is an $a\in\mathbf{R}$ which is not a left zero divisor then $a\mathbf{R}_0=\mathbf{R}_0$. So there is a permutation π of the indices such that $ax_i=x_{\pi(i)}$. Let $\pi^k=1$ $(k>0)$ then $a^kx_i=x_i$ for all i, implying that a^k is a left identity of **R**.

Suppose there is a c such that $ca=0$ then also $ca^k=0$ and $ca^kx=cx=0$ for all $x\in\mathbf{R}$. Hence $c=0$.

Because we have shown now that a is not a right zero divisor, either, there is an $l>0$ such that a^l is a right identity. But then $a^k=a^l$ is the unique, two-sided identity of **R**, a contradiction. □

THEOREM 3.5. *Any left admissible ring* **R** *has an identity.*

PROOF. Let **R** have no identity. Then we consider two cases.

First suppose that $\mathbf{Ann}^l(\mathbf{R})\neq\{0\}$. Let χ be any character of $\mathbf{R}^+$ and $a\in\mathbf{Ann}^l(\mathbf{R})\setminus\{0\}$.

Then ${}^{(a)}\chi(x)=\chi(ax)=\chi(0)=1$ for all $x\in\mathbf{R}$. Hence ${}^{(a)}\chi={}^{(0)}\chi$, which contradicts the left admissibility of **R**.

Secondly, if $\mathbf{Ann}^l(\mathbf{R})=\{0\}$ we can apply theorem 3.3. Let χ be a left admissible character of $\mathbf{R}^+$. Consider for an $a\in\mathbf{R}\setminus\{0\}$ the character ${}^{(a)}\chi$. By lemma 3.4 there is an element $b\in\mathbf{R}\setminus\{0\}$ such that $ab=0$. So $\mathbf{Ker}({}^{(a)}\chi)$ contains the non-zero right ideal $b\mathbf{R}$ and so ${}^{(a)}\chi$ is not admissible. But this implies $\chi\notin\Omega^l_\chi$, a contradiction.

Hence **R** has to have an identity. □

CONVENTION. From now on in this paper $\mathbf{R}$, with or without indices, denotes a finite ring *with identity.*

Theorem 3.3 implies that a ring $\mathbf{R}$ is admissible if $\mathbf{R}$ does not have any proper (left, right or two-sided) ideals or if $\mathbf{R}^+$ is cyclic. In the former case $\mathbf{R}$ must be a field and in the latter $\mathbf{R} \cong \mathbf{Z}_v$.

As a consequence of theorem 3.3 we have the following.

COROLLARY 3.6. *A character χ of $\mathbf{R}^+$ is left admissible if and only if $\mathbf{Ker}(\chi)$ does not contain minimal right ideals.*

It follows from the next theorem that left admissibility of $\mathbf{R}$ is completely determined by the internal structure of its right socle.

THEOREM 3.7. *Suppose that the ring $\mathbf{R}$ has exactly s minimal right ideals $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_s$, and define*

$$(1)\quad \begin{cases} \mathscr{P}^r(\mathbf{R}) = 1 - \sum_i \dfrac{1}{|\mathbf{I}_i|} + \sum_{i<j} \dfrac{1}{|\mathbf{I}_i + \mathbf{I}_j|} \\ \qquad - \sum_{i<j<k} \dfrac{1}{|\mathbf{I}_i + \mathbf{I}_j + \mathbf{I}_k|} + \cdots + (-1)^s \dfrac{1}{|\mathbf{I}_1 + \cdots + \mathbf{I}_s|} \end{cases}$$

then

1. *$\mathbf{R}$ is left admissible if and only if $\mathscr{P}^r(\mathbf{R}) > 0$;*
2. *$\mathbf{R}$ is not left admissible if and only if $\mathscr{P}^r(\mathbf{R}) = 0$.*

The number of left admissible characters in $\mathbf{R}$ is $\mathscr{P}^r(\mathbf{R}) \cdot |\mathbf{R}|$ and if $\mathbf{R}$ is left admissible then $\mathscr{P}^r(\mathbf{R}) \cdot |\mathbf{R}| = |\mathbf{U}(\mathbf{R})|$.

PROOF. We count all the characters of $\mathbf{R}^+$ which are not left admissible. All those characters φ have the property $\mathbf{I}_i \subset \mathbf{Ker}(\varphi)$ for some i. There are exactly $|\mathbf{R}|/|\mathbf{I}_{i_1} + \cdots + \mathbf{I}_{i_t}|$ characters φ for which $\mathbf{I}_{i_k} \subset \mathbf{Ker}(\varphi)$ for all k such that $1 \le k \le t$. By the principle of inclusion and exclusion the number of not left admissible characters of $\mathbf{R}^+$ is $(1 - \mathscr{P}^r(\mathbf{R}))|\mathbf{R}|$. This implies 1. and 2.

There remains to prove that the number of left admissible characters is $|\mathbf{U}(\mathbf{R})|$ in the case that $\mathbf{R}$ is left admissible. This can be done as follows.

Let ω be a left admissible character of $\mathbf{R}^+$. If $a \in \mathbf{U}$ then it is easily established that $\mathbf{Ker}({}^{(a)}\omega) = a^{-1}\mathbf{Ker}(\omega)$. Plainly this yields the fact that ${}^{(a)}\omega$ is admissible if $a \in \mathbf{U}$.

Let a be a non-unit. Then there exists an element $b \in \mathbf{R} \setminus \{0\}$ such that $a \,.\, b = 0$. Hence the proper right ideal $b\mathbf{R} \subset \mathbf{Ker}({}^{(a)}\omega)$. So ${}^{(a)}\omega$ is not left admissible. □

In the same way as $\mathscr{P}^r(\mathbf{R})$ is defined in formula (1) we define $\mathscr{P}^l(\mathbf{R})$ by just summing over the minimal *left* ideals of $\mathbf{R}$.

The next lemma addresses the question which characters of a left admissible ring are left admissible. Its proof can be found in the proof of theorem 3.7.

LEMMA 3.8. *Let* $\mathbf{R}$ *be a left admissible ring. Suppose that* ω *is a left admissible character. Then* ${}^{(a)}\omega$ *is left admissible if and only if* $a \in \mathbf{U}$.

THEOREM 3.9. *Let* $\mathbf{R}$ *be an admissible ring then every left or right admissible character is admissible.*

PROOF. $\mathbf{R}$ has by definition left and right admissible characters. Let χ be a left admissible and let ω be a right admissible one.

Then $\omega = {}^{(a)}\chi$ for some $a \in \mathbf{R}$ and $\chi = \omega^{(b)}$ for some $b \in \mathbf{R}$. Hence $\chi(x) = \chi(axb)$ for all $x \in \mathbf{R}$ implying

(2) $\quad x - axb \in \mathbf{Ker}(\chi)$ for all $x \in \mathbf{R}$.

If ω were not left admissible then $a \in \mathbf{R} \setminus \mathbf{U}$. Therefore there is an $a_0 \in \mathbf{R} \setminus \{0\}$ such that $aa_0 = 0$. If $r \in a_0\mathbf{R}$ then by (2), $r - arb = r \in \mathbf{Ker}(\chi)$ and so the proper right ideal $a_0\mathbf{R}$ belongs to $\mathbf{Ker}(\chi)$. But this contradicts the fact that χ is left admissible.

Therefore ω must also be left admissible. Along the same lines the proof of the theorem can be completed. □

LEMMA 3.10. *If* $\mathbf{R} = \mathbf{R}_1 \oplus \mathbf{R}_2$ *then* $\mathscr{P}^r(\mathbf{R}) = \mathscr{P}^r(\mathbf{R}_1) \cdot \mathscr{P}^r(\mathbf{R}_2)$.

PROOF. The general term of $\mathscr{P}^r(\mathbf{R})$ is $\mathrm{sign}(\mathbf{I})/|\mathbf{I}|$, where $\mathrm{sign}(\mathbf{I}) = +1$ if $\mathbf{I}$ is the sum of an even number of minimal right ideals, and -1 otherwise. Because $\mathbf{S}^r(\mathbf{R}) = \mathbf{S}^r(\mathbf{R}_1) \oplus \mathbf{S}^r(\mathbf{R}_2)$

$$\frac{\mathrm{sign}(\mathbf{I})}{|\mathbf{I}|} = \frac{\mathrm{sign}(\mathbf{I}_1)}{|\mathbf{I}_1|} \cdot \frac{\mathrm{sign}(\mathbf{I}_2)}{|\mathbf{I}_2|}$$

if $\mathbf{I} = \mathbf{I}_1 \oplus \mathbf{I}_2$ with $\mathbf{I}_i \in \mathbf{R}_i$. But this implies the lemma. □

THEOREM 3.11. *Suppose the finite ring* $\mathbf{R}$ *is the direct sum of the rings* $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_k$. *Then* $\mathbf{R}$ *is left admissible if and only if every* $\mathbf{R}_i$ *is left admissible.*

PROOF. The theorem is an easy consequence of lemma 3.10 and theorem 3.7. □

Theorem 3.11 can also be proved directly, without the use of the $\mathscr{P}^r$-function; for this we refer to the proof of theorem 3.2.13 in [1].

The results of this section show that the study of admissibility of rings is in fact the study of the socles of indecomposable rings. Before we study (semi)-simple rings and local rings we give in the next section first a few illustrations of the way how one can use the characterization of left admissibility given by theorem 3.7.

4. RINGS WITH A FEW MINIMAL RIGHT IDEALS

As a preliminary result we prove the following.

THEOREM 4.1. *If the right socle of the ring* $\mathbf{R}$ *is the direct sum of all minimal right ideals of* $\mathbf{R}$, *then* $\mathbf{R}$ *is left admissible.*

PROOF. Suppose $\mathbf{R}$ is not left admissible, then $\mathscr{P}^r(\mathbf{R})=0$. So if $|\mathbf{I}_i|=t_i$ for the minimal right ideals $\mathbf{I}_1,\ldots,\mathbf{I}_s$ we get:

$$1-\sum_i \frac{1}{t_i}+\sum_{i<j}\frac{1}{t_it_j}-\cdots+(-1)^s\frac{1}{t_1t_2\cdots t_s}=0.$$

But this is equivalent to $(t_1-1)(t_2-1)\cdots(t_s-1)=0$, a contradiction since $t_i\geq 2$ for all i. $\square$

THEOREM 4.2. *The following holds.*

1. *A ring with exactly one minimal right ideal is left admissible.*
2. *A ring with exactly two minimal right ideals is left admissible.*

PROOF. This is an immediate consequence of the preceding theorem. $\square$

THEOREM 4.3. *A ring with exactly three minimal right ideals is not left admissible if and only if the minimal right ideals are* $\{0,x\}$, $\{0,y\}$ *and* $\{0,x+y\}$ *for some well-chosen* $x,y\in\mathbf{R}$, *and in that case* $2\mid\gamma(\mathbf{R})$.

PROOF. Let $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ be the minimal right ideals of $\mathbf{R}$. First suppose that $\mathbf{I}_1\cap(\mathbf{I}_2\oplus\mathbf{I}_3)=(0)$, then $\mathbf{S}^r(\mathbf{R})=\mathbf{I}_1\oplus\mathbf{I}_2\oplus\mathbf{I}_3$, and $\mathbf{R}$ is left admissible by theorem 4.1.

Secondly suppose that $\mathbf{I}_1\subset\mathbf{I}_2\oplus\mathbf{I}_3$. It follows directly that $\mathbf{I}_i\subset\mathbf{I}_j\oplus\mathbf{I}_k$, where (i,j,k) is any permutation of $(1,2,3)$. But this yields

$$\mathbf{S}^r(\mathbf{R})=\mathbf{I}_1\oplus\mathbf{I}_2=\mathbf{I}_1\oplus\mathbf{I}_3=\mathbf{I}_2\oplus\mathbf{I}_3.$$

and so, with $|\mathbf{I}_i|=t_i$, we find $t_1t_2=t_1t_3=t_2t_3$, implying $t_1=t_2=t_3=t$.

$\mathbf{R}$ is not left admissible if and only if $\mathscr{P}^r(\mathbf{R})=0$ which is in this case equivalent to $(t-1)(t-2)=0$. So $t=2$, from which the first statement follows.

Evidently 2 is a zero divisor in $\mathbf{R}$ and so $2\mid\gamma(\mathbf{R})$. $\square$

THEOREM 4.4. *A ring with exactly four minimal right ideals is not left admissible if and only if*

1. *either*
   (a) $2\mid\gamma(\mathbf{R})$ *and*
   (b) *three of the four minimal right ideals of* $\mathbf{R}$ *have, for suitably chosen* $x,y\in\mathbf{R}$, *the form* $\{0,x\}$, $\{0,y\}$ *and* $\{0,x+y\}$; *in that case the fourth minimal right ideal* $\mathbf{I}$ *has the property*

$$\mathbf{S}^r(\mathbf{R})=\{0,x\}\oplus\{0,y\}\oplus\mathbf{I}$$

2. *or*
   (a) $3 \mid \gamma(\mathbf{R})$ *and*
   (b) *the minimal right ideals of* $\mathbf{R}$ *are, for suitably chosen* x *and* y *in* $\mathbf{R}$, $\{0, x, -x\}$, $\{0, y, -y\}$, $\{0, x+y, -x-y\}$ *and* $\{0, x-y, -x+y\}$.

PROOF. Let $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ denote the minimal right ideals of $\mathbf{R}$ and let $|\mathbf{I}_i| = s_i$. If the right socle $\mathbf{S}^r(\mathbf{R})$ is the direct sum of the four minimal right ideals then $\mathbf{R}$ is left admissible by theorem 4.1. Hence without loss of generality we may restrict ourselves to the following three cases.

a. $\mathbf{S}^r(\mathbf{R}) = \mathbf{I}_1 \oplus \mathbf{I}_2 \oplus \mathbf{I}_3$ and $\mathbf{I}_4 \cap (\mathbf{I}_i \oplus \mathbf{I}_j) = (0)$ for all $1 \le i < j \le 3$.

b. $\mathbf{S}^r(\mathbf{R}) = \mathbf{I}_1 \oplus \mathbf{I}_2 \oplus \mathbf{I}_3$ and $\mathbf{I}_4 \subset \mathbf{I}_1 \oplus \mathbf{I}_2$.

c. $\mathbf{S}^r(\mathbf{R}) = \mathbf{I}_1 \oplus \mathbf{I}_2$ and $\mathbf{I}_3, \mathbf{I}_4 \subset \mathbf{I}_1 \oplus \mathbf{I}_2$.

In case **a.** it follows that $\mathbf{I}_1 \oplus \mathbf{I}_2 \oplus \mathbf{I}_3 = \mathbf{I}_1 \oplus \mathbf{I}_2 \oplus \mathbf{I}_4$ etc., from which we get $s_1 = s_2 = s_3 = s_4 = s$. $\mathscr{P}^r(\mathbf{R}) = 0$ is now equivalent to $s^3 - 4s^2 + 6s - 3 = 0$. For $s \in \mathbf{N} \setminus \{0, 1\}$ this is not possible and so $\mathbf{R}$, if existent, must be left admissible in this case.

In case **b.** also $\mathbf{I}_1 \subset \mathbf{I}_2 \oplus \mathbf{I}_4$ and $\mathbf{I}_2 \subset \mathbf{I}_1 \oplus \mathbf{I}_4$, hence $s_1 = s_2 = s_4 = s \ge 2$. Let $s_3 = t \ge 2$. If we suppose that $\mathbf{R}$ is not left admissible then $\mathscr{P}^r(\mathbf{R}) = 0$ is equivalent to $(s-1)(s-2)(t-1) = 0$. Hence the results of 1. follow directly.

In case **c.** clearly $s_1 = s_2 = s_3 = s_4 = s \ge 2$ holds. The criterion for being not left admissible now implies $(s-1)(s-3) = 0$, from which 2. follows. □

EXAMPLE 4.5. Consider the following three rings.

1. $\mathbf{R}_1 = \mathbf{GF}(2)[x, y]/(x^2, y^2, xy, xy - yx)$,
2. $\mathbf{R}_2 = \mathbf{GF}(2)[x, y]/(x^2, y^2, xy - yx)$,
3. $\mathbf{R}_3 = \mathbf{GF}(2)[x, y]/(x^2, y^2, xy)$.

As noticed in example 3.2 the ring $\mathbf{R}_1$ has 3 minimal ideals, viz. $\{0, x\}$, $\{0, y\}$ and $\{0, x+y\}$, and therefore according to theorem 4.3 the ring is not admissible.

By theorem 4.2 the ring $\mathbf{R}_2$ is admissible because $\mathbf{R}_2$ has the unique minimal ideal $\mathbf{R}_2 xy = \{0, xy\}$. There are 8 characters χ such that $\chi(xy) = -1$ and these are the admissible ones. Notice that $|\mathbf{U}(\mathbf{R}_2)| = 8$, which is in accordance with theorem 3.7.

The ring $\mathbf{R}_3$ is neither left nor right admissible. We can think of at least three ways to prove this.

First one can prove the non-admissibility directly. Suppose φ were a left admissible character then

- $\varphi(x) = -1$, because $\{0, x\}$ is a right minimal ideal and also
- $\varphi(yx) = -1$, because $\{0, yx\}$ is a (two-sided) minimal ideal.

However, then $\varphi(x + yx) = 1$ and this would imply that $\mathbf{Ker}(\varphi)$ contained the right minimal ideal $\{0, x + yx\}$ and we had a contradiction. Because $\{0, y\}$ is a left minimal ideal we can prove in the same way that $\mathbf{R}$ cannot be right admissible either.

Secondly theorem 4.3 can be used, because of the structure of the left and the right socle of $\mathbf{R}$.

Finally one can prove that $\mathbf{R}$ is not admissible by using theorem 8.3. $\diamond$

5. SEMI-SIMPLE RINGS

REMARK 5.1. At first we introduce some notations.

1. $\mathbf{M}_{s,t}(q)$ is the set of all $(s \times t)$-matrices over the field $\mathbf{GF}(q)$; the all-zero matrix is denoted by 0. It is well-known that a finite ring is a simple ring if and only if that ring is isomorphic to a full matrix ring $\mathbf{M}_{m,m}(q)$.
2. If $m \in \mathbf{N} \setminus \{0\}$ and q is a prime power then if $A \in \mathbf{M}_{1,m}(q)$ we let

   $\mathbf{J}_A = \{X \cdot A \mid X \in \mathbf{M}_{m,1}(q)\}$.
3. If $m \in \mathbf{N} \setminus \{0\}$ then
   (a) $E_{ij} \in \mathbf{M}_{m,m}(q)$ is the matrix with the (i,j) entry 1 and all other entries 0;
   (b) $F_i \in \mathbf{M}_{m,1}(q)$ is the matrix with the i-th entry 1 and all other entries 0. $\diamond$

As a preparation of the main theorem of this section we prove first the following lemma.

LEMMA 5.2. *If $\mathbf{R}$ is the simple ring $\mathbf{M}_{m,m}(q)$, where q is the prime power p^r, then the following holds.*

1. *A left ideal $\mathbf{J}$ of $\mathbf{R}$ is minimal if and only if there is an $A \in \mathbf{M}_{1,m}(q) \setminus \{0\}$ such that $\mathbf{J} = \mathbf{J}_A$.*
2. *If $A \in \mathbf{M}_{1,m}(q) \setminus \{0\}$ and R is arbitrarily chosen in $\mathbf{M}_{m,1}(q) \setminus \{0\}$ we have*
   (a) $\mathbf{J}_A = \mathbf{R}(R \cdot A)$;
   (b) *the matrices $F_i \cdot A \in \mathbf{M}_{m,m}(q)$ $(i = 1, 2, \ldots, m)$ form a basis for $\mathbf{J}_A$;*
   (c) *if $\mathbf{GF}(q) = \mathbf{GF}(p)(\theta)$, and $A = A_0 + \theta A_1 + \cdots + \theta^{r-1} A_{r-1}$ with $A_i \in \mathbf{M}_{1,m}(p)$ and if $A_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,m})$ then*

   $$F_i \cdot A = \sum_{k=1}^{m} (a_{0,k} + a_{1,k}\theta + \cdots + a_{r-1,k}\theta^{r-1}) E_{ik}.$$
3. *There are $(q^m - 1)/(q - 1)$ minimal left ideals in $\mathbf{R}$.*

PROOF. At first remark that $\mathbf{J}_A$ is a left ideal since if $S_1, S_2 \in \mathbf{M}_{m,1}(q)$

$$S_1 \cdot A - S_2 \cdot A = (S_1 - S_2) \cdot A \in \mathbf{J}_A$$

and if $M \in \mathbf{R}$ and $S \in \mathbf{M}_{m,1}(q)$ then $M \cdot (S \cdot A) = (M \cdot S) \cdot A \in \mathbf{J}_A$.

Plainly if $R, S \in \mathbf{M}_{m,1}(q)$, and $R \neq 0$, then there is an $M \in \mathbf{R}$ such that $M \cdot R = S$. This implies that $\mathbf{J}_A$ is a principal left ideal generated by $R \cdot A$, where R is any element of $\mathbf{M}_{m,1}(q) \setminus \{0\}$.

Now, the fact that the matrices $F_i \cdot A$ $(i = 1, 2, \ldots, m)$ form a basis for $\mathbf{J}_A$ is evident, whereas 2(c) is just another way of representing the matrices $F_i \cdot A$.

If $c \in \mathbf{GF}(q) \setminus \{0\}$ then $\mathbf{J}_A = \mathbf{J}_{cA}$, since $cX \cdot A = X \cdot (cA)$ for any $X \in \mathbf{M}_{m,1}(q)$ and because $\mathbf{J}_A \cap \mathbf{J}_B = \{0\}$ if A and B are independent over $\mathbf{GF}(q)$ there are $(q^m - 1)/(q - 1)$ left ideals of the form $\mathbf{J}_A$ in $\mathbf{R}$.

Let $\mathbf{I}$ be any non-zero left ideal of $\mathbf{R}$. Clearly if $Y \in \mathbf{R} \setminus \{0\}$ and Y_j is the j-th row of Y then $E_{ij} \cdot Y = F_i \cdot Y_j$. Hence if $Y \in \mathbf{I} \setminus \{0\}$ and A is a non-zero row of Y then because $\mathbf{I}$ is a left ideal $\mathbf{J}_A \subset \mathbf{I}$. This implies 1. and the proof of the lemma is complete. □

THEOREM 5.3. *Any finite semi-simple ring* $\mathbf{R}$ *is admissible.*

PROOF. By theorem VIII.4 in [5] $\mathbf{R}$ is the direct sum of simple rings. So by theorem 3.11 we only have to consider a simple ring $\mathbf{R}$.

Let $\mathbf{R} = \mathbf{M}_{m,m}(q)$, and $q = p^r$ while $\mathbf{GF}(q) = \mathbf{GF}(p)(\theta)$ with

$$\theta^r = c_{r-1}\theta^{r-1} + \cdots + c_1\theta + c_0,$$

$c_i \in \mathbf{GF}(p)$. Note $c_0 \neq 0$, necessarily.

As is easily checked the elements $\theta^k E_{ij}$ of $\mathbf{R}$ with $1 \le i, j \le m$ and $0 \le k \le r-1$ form a basis for $\mathbf{R}$ over $\mathbf{GF}(p)$ and so if χ is any character of $\mathbf{R}^+$, then χ is completely determined by the numbers $\chi(\theta^k E_{ij})$, $1 \le i, j \le m$ and $0 \le k \le r-1$ and it is immediate that these numbers are p-th roots of unity.

Let ζ be a p-th root of unity $\neq 1$ then we define the character ω of $\mathbf{R}^+$ as follows:

$$\omega(\theta^k E_{ij}) = \begin{cases} \zeta & \text{if } (i,j,k) = (i,i,0), \\ 1 & \text{otherwise.} \end{cases}$$

We shall show that ω is a right admissible character of $\mathbf{R}^+$.

Suppose there is an $A \in \mathbf{M}_{1,m}(q)$ such that $\mathbf{J}_A \in \mathbf{Ker}(\omega)$ then in the setting of lemma 5.2, item 2(c) we have $1 = \omega(F_i \cdot A) = \zeta^{a_{0,i}}$ for $i = 1, 2, \ldots, m$, and this yields $A_0 = 0$. So $A = A_1\theta + \cdots + A_{r-1}\theta^{r-1}$. Now

$$\theta A = c_0 A_{r-1} + c_1 A_{r-1}\theta + (A_1 + c_2 A_{r-1})\theta^2 + \cdots + (A_{r-2} + c_{r-1}A_{r-1})\theta^{r-1}.$$

Hence, because $\theta(F_i \cdot A) = F_i \cdot (\theta A)$ we have now $1 = \omega(\theta(F_i \cdot A)) = \zeta^{c_0 a_{r-1,i}}$ for $i = 1, 2, \ldots, m$, and because $c_0 \neq 0$, we find $A_{r-1} = 0$ and A reduces to

$$A = A_1\theta + \cdots + A_{r-2}\theta^{r-2}.$$

Successively considering $\theta^j(F_i \cdot A)$ we find $A_{r-j} = 0$ for $j = 2, 3, \ldots, r-1$ and so in the end we have proved that $A = 0$ if $\mathbf{J}_A \subset \mathbf{Ker}(\omega)$. So no left minimal ideal of $\mathbf{R}$ is contained in $\mathbf{Ker}(\omega)$ and we may conclude that ω is right admissible.

Because the map $B \to B^T$ of $\mathbf{R}$ on $\mathbf{R}$ is an anti-automorphism of $\mathbf{R}$ theorem 8.1 implies that $\mathbf{R}$ is admissible. □

As one easily checkes the ring $\mathbf{M}_{m,m}(q)$ provides an example of a ring that is left admissible, but its right socle (which is the ring itself) is *not* the direct sum of *all* minimal right ideals; compare this with theorem 7.1.

6. LOCAL RINGS

We need some preparatory results.

THEOREM 6.1. *Let* $\mathbf{R}$ *be a local ring then the following holds.*

1. $\mathbf{S}^r(\mathbf{R}) = \mathbf{Ann}^l(\mathbf{Rad}(\mathbf{R}))$.
2. $\mathbf{S}^r(\mathbf{R})$ *can be made into a vectorspace over* $\mathbf{R}/\mathbf{Rad}(\mathbf{R})$, *such that the* 1*-dimensional subspaces are just the minimal right ideals of* $\mathbf{R}$.

PROOF. Let $\mathbf{A} = \mathbf{Ann}^l(\mathbf{Rad}(\mathbf{R}))$. By lemma 2.2, $\mathbf{S}^r \subset \mathbf{A}$.

Conversely if $x \in \mathbf{A}$ then $xz\mathbf{R} = (0)$ if $z \in \mathbf{Rad}$ and $xz\mathbf{R} = x\mathbf{R}$ if $z \in \mathbf{R} \setminus \mathbf{Rad}$. So $x\mathbf{R}$ is minimal right ideal. As a result $\mathbf{A} \subset \mathbf{S}^r$. This proves 1.

If $x \in \mathbf{S}^r$ and $z \in \mathbf{R}$ then we have $x(z + \mathbf{Rad}) = xz$. But this implies the second assertion. □

LEMMA 6.2. *Let* $\mathbf{V}$ *be a t-dimensional vectorspace over a finite field* $\mathbf{F}$. *Let* χ *be a non-principal character of* $\mathbf{V}^+$, *the additive group of* $\mathbf{V}$. *Then* $\mathbf{Ker}(\chi)$ *contains one and only one* $(t-1)$*-dimensional subspace of* $\mathbf{V}$.

PROOF. Suppose $\mathbf{F} = \mathbf{GF}(p^r)$ and let θ be a primitive element of $\mathbf{F}$. Then $\mathbf{F} = \mathbf{K}(\theta)$, where $\mathbf{K}$ denotes the prime field of $\mathbf{F}$.

$\mathbf{V}$ is also an rt-dimensional vectorspace over $\mathbf{K}$. Because χ is a non-principal character of $\mathbf{V}^+$, $\mathbf{Ker}(\chi)$ is an $(rt-1)$-dimensional $\mathbf{K}$-subspace of $\mathbf{V}$. This can be seen as follows.

The group $\mathbf{V}^+$ is an elementary p-group, and so there is a primitive p-th root of unity ζ such that for all $x \in \mathbf{V}^+$ there is an $a \in \mathbf{K}$ with $\chi(x) = \zeta^a$. Connected with χ there is a linear functional L_χ of $\mathbf{V}$ into $\mathbf{K}$ such that if $\chi(x) = \zeta^a$ then $L_\chi(x) = a$. Plainly $\mathbf{Ker}(\chi)$ is the subspace of $\mathbf{V}$ over $\mathbf{K}$ characterized by $L_\chi(x) = 0$. The dimension of such a space is $rt-1$.

Since $x \to \theta x$ is a linear map on $\mathbf{V}$ ($\mathbf{V}$ considered as a vectorspace over $\mathbf{K}$) it follows that $\theta^i \cdot \mathbf{Ker}(\chi)$ is for every i also an $(rt-1)$-dimensional $\mathbf{K}$-subspace of $\mathbf{V}$. Now let $\mathbf{Y} = \mathbf{Ker}(\chi) \cap \theta \cdot \mathbf{Ker}(\chi) \cap \cdots \cap \theta^{r-1} \cdot \mathbf{Ker}(\chi)$ then the dimension d of $\mathbf{Y}$ satisfies $rt - r \le d \le rt - 1$. But this and $\theta \cdot \mathbf{Y} = \mathbf{Y}$ implies that $\mathbf{Y}$ is a $(t-1)$-dimensional $\mathbf{F}$-subspace contained in $\mathbf{Ker}(\chi)$. Because $\mathbf{V} \ne \mathbf{Ker}(\chi)$ the uniqueness of $\mathbf{Y}$ follows directly. □

EXAMPLE 6.3. Let $\mathbf{V}$ be a 2-dimensional vectorspace over $\mathbf{GF}(4)$. We represent $\mathbf{GF}(4)$ as $\mathbf{GF}(4) = \{0, 1, \theta, 1+\theta\}$ with $1 + \theta = \theta^2$.

Let (x, y) be a basis of $\mathbf{V}$ over $\mathbf{GF}(4)$. Let χ be the character of $\mathbf{V}^+$ such that $\chi(x) = \chi(\theta x) = \chi(y) = \chi(\theta y) = -1$. Clearly

$$\mathbf{Ker}(\chi) \cap \theta \cdot (\mathbf{Ker}(\chi)) = \{0, x+y, \theta x + \theta y, (1+\theta)x + (1+\theta)y\}$$

is 1-dimensional over $\mathbf{GF}(4)$, as it should. ◇

LEMMA 6.4. *Let* $\mathbf{R}$ *be a local ring. Then* $\mathbf{R}$ *is left admissible if and only if* $\mathbf{R}$ *has exactly one minimal right ideal.*

PROOF. If $\mathbf{R}$ has exactly one minimal right ideal then plainly $\mathbf{R}$ is left admissible.

Suppose **R** has more than one minimal right ideal then, according to theorem 6.1, the socle $\mathbf{S}^r(\mathbf{R})$ of **R** can be considered as a k-dimensional vectorspace over the field **R/Rad** for certain $k \geq 2$.

Let χ be any non-principal character of **R** then the restriction χ' of χ to $\mathbf{S}^r(\mathbf{R})$ is a character of $\mathbf{S}^r(\mathbf{R})^+$. By lemma 6.2 there is a $(k-1)$-dimensional subspace **Y** in $\mathbf{S}^r(\mathbf{R})$ contained in $\mathbf{Ker}(\chi')$. **Y** itself contains 1-dimensional subspaces, because $k \geq 2$. These 1-dimensional subspaces are minimal right ideals in **R**, according to theorem 6.1, and therefore there are ideals contained in $\mathbf{Ker}(\chi)$. This implies that **R** is not left admissible, because χ was arbitrarily chosen. This proves the lemma. □

The result of lemma 6.4 elucidates once more why the ring $\mathbf{R}_1$ considered in example 4.5 is not admissible: the local ring $\mathbf{R}_1$ has more than one minimal ideal.

THEOREM 6.5. *Let* **R** *be a local ring then* **R** *is admissible if and only if* **R** *has only one minimal right and only one minimal left ideal.*

In that case the two one-sided minimal ideals coincide, $\mathbf{S}^r = \mathbf{S}^l$ *and* $\mathbf{S}^r$ *is the only minimal (two-sided) ideal.*

PROOF. The first part of the theorem follows from lemma 6.4.

If **R** is admissible then obviously $\mathbf{S}^r$ is the only minimal right ideal and $\mathbf{S}^l$ is the only minimal left ideal. So $|\mathbf{S}^r| = |\mathbf{S}^l| = |\mathbf{R}/\mathbf{Rad}|$ by theorem 6.1. But $\mathbf{S}^r$ is a two-sided ideal and $\mathbf{S}^r = \mathbf{S}^l$ easily follows. □

Rings as described in theorem 6.5 exist. The ring $\mathbf{R}_2$ in example 4.5 is a commutative one and the local ring $\mathbf{GF}(q)[x,y]/(xy, x^2-yx, y^2-yx)$ is an instance of a non-commutative ring with only one minimal two-sided ideal, viz. (yx).

In view of lemma 6.4 the next question is of importance in connection with the question posed at the beginning of section 8.

QUESTION: Does there exist local rings with one minimal right ideal but with more than one minimal left ideal?

THEOREM 6.6. *Let* **R** *be a ring which is the direct sum of local rings. Then the following statements are equivalent.*

1. **R** *is left admissible.*
2. $\mathbf{S}^r(\mathbf{R})$ *is the direct sum of all minimal right ideals.*
3. *Every local direct summand of* **R** *has exactly one minimal right ideal.*

PROOF. In view of lemma 6.4 and theorem 3.11 the assertions 1. and 3. are equivalent.

2. implies 1. by theorem 4.1. 1. implies 3. and this implies 2. This completes the proof of the theorem. □

EXAMPLE 6.7. We give now a class of local rings which are neither left nor right admissible.

Let $m \geq 3$ and let $\mathbf{R}$ be the ring of the upper triangular $(m \times m)$-matrices with entries from the field $\mathbf{GF}(q)$, such that the entries on the main diagonal all are equal. Under these conditions $\mathbf{R}$ is a local ring. The radical of $\mathbf{R}$ consists of all upper triangular matrices whose entries on the main diagonal are 0. $\mathbf{R}$ is non-commutative, since $m \geq 3$. We shall show that $\mathbf{R}$ is neither left admissible nor right admissible. We shall use the matrices E_{ij} introduced in remark 5.1.

It is trivial to show that

$$\mathbf{I}_i^l = \{\lambda E_{1i} \mid \lambda \in \mathbf{GF}(q)\} \text{ for } i = 2, 3, \dots, m$$

are different, minimal, left ideals in $\mathbf{R}$ and that

$$\mathbf{I}_j^r = \{\lambda E_{jm} \mid \lambda \in \mathbf{GF}(q)\} \text{ for } j = 1, 2, \dots, m-1$$

are different, minimal, right ideals in $\mathbf{R}$.

Hence by lemma 6.4 the ring $\mathbf{R}$ is neither left nor right admissible. ◇

7. COMMUTATIVE RINGS

THEOREM 7.1. *Let* $\mathbf{R}$ *be a commutative ring. Then the following statements are equivalent.*

1. $\mathbf{R}$ *is admissible.*
2. *The socle* $\mathbf{S}(\mathbf{R})$ *is the direct sum of all minimal ideals.*
3. *Every local direct summand of* $\mathbf{R}$ *has exactly one minimal ideal.*

PROOF. The structure theorem for finite commutative rings with identity [5, Theorem VI.2] states that such a commutative ring decomposes uniquely (up to the order of the summands) as a direct sum of local rings. So the theorem is a direct consequence of theorem 6.6. □

COROLLARY 7.2. *Any commutative principal ideal ring* $\mathbf{R}$ *is admissible.*

PROOF. Obviously the local direct summands of $\mathbf{R}$ are chain rings. By theorem 7.1 this implies the admissibility of $\mathbf{R}$. □

Consider the commutative ring $\mathbf{R} = \mathbf{Z}_6[x,y]/(x^2, y^2, xy - yx)$. $\mathbf{R}$ is the direct sum of two local rings $\mathbf{Z}_2[x,y]/(x^2, y^2, xy - yx)$ and $\mathbf{Z}_3[x,y]/(x^2, y^2, xy - yx)$ both containing one minimal ideal. Also $\mathbf{S}(\mathbf{R}) = (xy) = (2xy) \oplus (3xy)$.

According to theorem 7.1 the ring $\mathbf{R}$ is admissible. In view of corollary 7.2 note that $\mathbf{R}$ is not a principal ideal ring, which is admissible nevertheless.

8. FINAL REMARKS

The question whether there exist rings which are left but not right admissible is still open. In this respect we have the following result.

THEOREM 8.1. *If a ring* $\mathbf{R}$ *has an anti-automorphism then the following are equivalent.*

1. $\mathbf{R}$ *is admissible.*
2. $\mathbf{R}$ *is left admissible.*
3. $\mathbf{R}$ *is right admissible.*

PROOF. If α is an anti-automorphism of $\mathbf{R}$ then $\alpha(\mathbf{S}^r)=\mathbf{S}^l$ implying $\mathscr{P}^r(\mathbf{R})=\mathscr{P}^l(\mathbf{R})$. □

We end this paper with some remarks on the number of orbits of a group of units in a non-commutative ring. We consider the following situation.

Let $\mathbf{R}$ be a non-commutative ring with identity. Let $\mathbf{M}$ be a fixed subgroup of $\mathbf{U}$. The elements $m\in\mathbf{M}$ will be regarded as inducing automorphisms of $\mathbf{R}^+$ in two ways, either by considering them as left multiplications defining $m^l(r)=mr$ or as right multiplications defining $m^r(r)=rm$ with $r\in\mathbf{R}$; the resulting groups of automorphisms of $\mathbf{R}^+$ will be denoted by $\mathbf{M}^l$ and $\mathbf{M}^r$, respectively.

Let $r_0\in\mathbf{R}$ then $\{mr_0 \mid m\in\mathbf{M}\}$ is called an *orbit* of $\mathbf{M}^l$. In the same way an *orbit* of $\mathbf{M}^r$ is defined. Denote the number of orbits of $\mathbf{M}^l$ (acting on $\mathbf{R}^+$) by n^l and the number of orbits of $\mathbf{M}^r$ (acting on $\mathbf{R}^+$) by n^r.

EXAMPLE 8.2. Consider the ring $\mathbf{R}_3=\mathbf{Z}_2[x,y]/(x^2,y^2,xy)$ given in example 4.5. We shall show that this ring is an example of a ring with a subgroup $\mathbf{M}$ of $\mathbf{U}$ such that $n^l\neq n^r$.

$\mathbf{R}_3$ is a local ring of characteristic 2.

$$\mathbf{Rad}(\mathbf{R}_3)=\{0,x,y,x+y,yx,x+yx,y+yx,x+y+yx\},$$

whereas $\mathbf{U}=1+\mathbf{Rad}(\mathbf{R}_3)$.

We take $\mathbf{M}=\{1,1+x\}$. $\mathbf{M}^l$ and $\mathbf{M}^r$ both have 4 orbits in $\mathbf{U}$. It is easily seen that $(1+x)r=r$ for all $r\in\mathbf{Rad}(\mathbf{R}_3)$. This implies that $n^l=12$. On the other hand the orbits of $\mathbf{M}^r$ in $\mathbf{Rad}(\mathbf{R}_3)$ are

$$\{0\},\{x\},\{y,y+yx\},\{x+y,x+y+yx\},\{yx\},\{x+yx\},$$

implying $n^r=10$.

Remark that for the group $\mathbf{U}$ the number of left orbits and the number of right orbits are both equal to 7. ◇

Using the theory of association schemes the following theorem can be proved; cf. [1, Theorem 3.3.1] and [3].

THEOREM 8.3. *Let* $\mathbf{R}$ *be a left admissible ring and let* $\mathbf{M}$ *be any subgroup of* $\mathbf{U}(\mathbf{R})$, *then* $n^r\geq n^l$ *holds.*

Let $\mathbf{R}$ *be an admissible ring and let* $\mathbf{M}$ *be any subgroup of* $\mathbf{U}(\mathbf{R})$, *then* $n^r=n^l$ *holds.*

According to theorem 8.3 the ring $\mathbf{R}_3$ used in the examples 4.5 and 8.2 cannot be left admissible, because for the group $\mathbf{M}$ used in example 8.2 $n^r < n^l$. In the same way, using the group $\{1, 1+y\}$, one easily shows that $\mathbf{R}_3$ cannot be right admissible either.

There are rings which are not admissible for which the condition $n^l = n^r$ still holds for all subgroups of $\mathbf{U}$. The ring $\mathbf{R}_0$ considered in example 2.4 is such a ring, as one easily checks. The background for the fact that $n^l = n^r$ holds for the groups of units in $\mathbf{R}_0$ is provided by the following evident lemma.

LEMMA 8.4. *Let* $\mathbf{R}$ *be a ring and* $\mathbf{M}$ *a subgroup of* $\mathbf{U}(\mathbf{R})$. *If* $\mathbf{R}$ *has an anti-automorphism* α *such that* $\alpha(\mathbf{M}) = \mathbf{M}$ *then* $n^l = n^r$ *holds for* $\mathbf{M}$.

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